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A conjecture on Hubbard–Stratonovich transformations for the Pruisken–Schäfer parameterizations of real hyperbolic domains

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Abstract

Rigorous justification of the Hubbard–Stratonovich transformation for the Pruisken–Schäfer type of parametrizations of real hyperbolic $O(m, n)$-invariant domains remains a challenging problem. We show that a naive choice of the volume element invalidates the transformation and put forward a conjecture about the correct form which ensures the desired structure. The conjecture is supported by a complete analytic solution of the problem for groups $O(1, 1)$ and $O(2, 1)$, and by a method combining analytical calculations with a simple numerical evaluation of a two-dimensional integral in the case of the group $O(2, 2)$.

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1. Introduction and formulation of the conjecture

For more than two decades, the nonlinear σ-model methodology has been widely applied to studies of single electron motions in disordered and chaotic mesoscopic systems [1, 2]. The method was pioneered by Wegner [3] and further developed by Wegner and Schäfer [4], and Pruisken and Schäfer [5] in the framework of the replica method used to reduce one-particle Hamiltonians with microscopic disorder to a nonlinear σ-model. In the early 1980s, Efetov [6] introduced the supersymmetric variant of the method which avoided the problematic replica trick and directly led to the supermatrix version of the nonlinear σ-model. Since then this latter nonlinear σ-model has also been successfully applied to a variety of problems in the framework of random matrix approach to chaotic scattering [7, 8], quantum chromodynamics [9], as well as a few other fields of physics.

A standard derivation of the nonlinear σ-models requires to use at some point the so-called Hubbard–Stratonovich transformation:

$$C_n\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\hat{A}^2} = \int \mathcal{D}\hat{R}\, \exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A}\right), \tag{1.1}$$

where $\hat{R}$ and $\hat{A}$ are $n \times n$ matrices and C_n is a normalization factor independent of the matrix $\hat{A}$. When matrices $\hat{R}$ and $\hat{A}$ are, for example, complex Hermitian, the volume element can be chosen as $\mathcal{D}\hat{R} \propto \prod_{i\leqslant j} \mathrm{d}[\mathrm{Re}\, R_{ij}]\mathrm{d}[\mathrm{Im}\, R_{ij}]$, and the above integral amounts to a product of standard Gaussian integrals over independent degrees of freedom, the identity (1.1) following immediately. The same method works obviously for the real symmetric matrices. On the other hand, in these simple cases we also have a freedom to go to 'polar' coordinates in the standard way. For example, for the complex Hermitian case [10]

$$\hat{R} = \hat{U}^{-1}\mathrm{diag}(p_1, \ldots, p_n)\hat{U}, \qquad \mathcal{D}R \propto \mathrm{d}\mu_H(U)\mathrm{d}P\,\Delta^2[\hat{P}], \tag{1.2}$$

where $\hat{U} \in U(n)$ is a unitary matrix of eigenvectors and $\hat{P} = \mathrm{diag}(p_1, \ldots, p_n)$ is the real diagonal matrix of the associated eigenvalues of $\hat{R}$, with $\mathrm{d}\mu_H(U)$ being the corresponding invariant Haar measure on the unitary group and $\Delta[\hat{P}] = \prod_{i<j}(p_j - p_j)$ standing for the Vandermonde determinant factor. Similarly, for the real symmetric matrices

$$\hat{R} = \hat{O}^{-1}\hat{P}\hat{O}, \qquad \mathcal{D}R \propto \mathrm{d}\mu_H(O)\mathrm{d}P|\Delta[\hat{P}]|, \tag{1.3}$$

with $\hat{O} \in O(n)$ being an orthogonal matrix.

In the problems of interest in electronic transport and random matrix theory the structure of the matrices $\hat{R}$ and $\hat{A}$ is however restricted by the underlying symmetries of the system, and is rather non-trivial, see [11] for a review. For the simplest choice of the disordered Hamiltonian corresponding to a system with broken time-reversal symmetry, one of the legitimate choices of the integration domain for R is due to Schäfer and Wegner [4]:

$$\hat{R} = \lambda\hat{T}\hat{T}^\dagger + \mathrm{i}\hat{P}, \tag{1.4}$$

where the matrices $\hat{T}$ must be chosen in the pseudounitary group: $\hat{T} \in U(n_1, n_2)$. The matrices $\hat{P}$ are Hermitian block-diagonal: $\hat{P} = \mathrm{diag}\big(\hat{P}_{n_1}, \hat{P}_{n_2}\big) = \hat{P}^\dagger$, and $\lambda > 0$ is an arbitrary positive number. For Hamiltonians respecting time-reversal symmetry the integration domain $\hat{R}$ is essentially of the same form, but with matrices $\hat{P}$ real symmetric block-diagonal and the matrices $\hat{T}$ taken as elements of the real pseudoorthogonal group: $\hat{T} \in O(n_1, n_2)$.

Although the Schäfer–Wegner parameterization of the integration manifold is correct, an accurate verification of the main formula (1.1) is not at all trivial and was provided only recently [11]. Actually, this type of parameterization has never been widely used in the physical literature. Instead, an alternative parameterization due to Pruisken and Schäfer [5] has been assumed, tacitly or explicitly, in the vast majority of applications:

$$\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}, \qquad \mathcal{D}R = \mathrm{d}\mu_H(T)\,\mathrm{d}P_1\,\mathrm{d}P_2\Delta^2[\hat{P}]. \tag{1.5}$$

Here we assumed the case of broken time-reversal symmetry, $\hat{T} \in U(n_1, n_2)$ and $\hat{P} = \mathrm{diag}\big(\hat{P}_{n_1}, \hat{P}_{n_2}\big)$, with $\hat{P}_{n_1}$ and $\hat{P}_{n_2}$ being real diagonal, $\mathrm{d}\mu_H(T)$ being the invariant Haar measure on the pseudounitary group and $\Delta[\hat{P}] = \prod_{i<j}(p_j - p_j)$ is the Vandermonde determinant factor. Apparently, this parameterization is a complete analogue of that in formula (1.2), specified for the pseudounitary symmetry.

Similarly, one expects that a natural analogue of (1.3) for the preserved time-reversal Hamiltonians and emerging real-hyperbolic domain should be

$$\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}, \qquad \mathcal{D}R = \mathrm{d}\mu_H(T)\mathrm{d}P_1\,\mathrm{d}P_2|\Delta[\hat{P}]|, \tag{1.6}$$

where the time $\hat{T} \in O(n_1, n_2)$ is the corresponding pseudo-orthogonal matrices.

To the best of our knowledge, the validity of the Hubbard–Stratonovich transformation with the Pruisken–Schäfer choice of the integration domain has not been carefully checked, but rather taken for granted. In fact, the simplest version of the 'deformation of contour' argument used to verify the transformation for the Schäfer–Wegner domain fails for the Pruisken–Schäfer choice [11], and this raised legitimate doubts on its validity in general, see also [12].

Given the widespread use of the Pruisken–Schäfer parameterization, as well as known technical advantages of working with it in some microscopic models, the situation clearly calls for further analysis. To this end, a rigorous proof of the validity of the Hubbard–Stratonovich transformation for the general pseudounitary Pruisken–Schäfer domain (1.5) was given for the first time by one of the authors [13]. In the same paper, a variant of the Hubbard–Stratonovich transformation for disordered systems with an additional chiral symmetry was also provided.

On the other hand, the problem of verifying Hubbard–Stratonovich transformation for the general real pseudoorthogonal Pruisken–Schäfer domain (1.6) turned out to be much more challenging due to serious technical difficulties to be discussed later on in the text of the paper. Only the simplest, yet non-trivial case $O(1, 1)$ was managed successfully in [13], and we summarize the results of that study below. The integration domain on the right-hand side of equation (1.1) is given explicitly by

$$\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}, \tag{1.7}$$

where

$$\hat{T} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \in \frac{O(1,1)}{O(1)\times O(1)} \qquad \text{and} \qquad \hat{P} = \operatorname{diag}(p_1, p_2). \tag{1.8}$$

The matrices $\hat{A}$ in equation (1.1) have the following form:

$$\hat{A} = \begin{pmatrix} a_1 & -a \\ a & -a_2 \end{pmatrix}, \qquad \text{with} \quad a_1 > 0, \quad a_2 > 0, \quad |a| < \sqrt{a_1 a_2}. \tag{1.9}$$

As has been shown in [13], the desirable form (1.1) of the Hubbard–Stratonovich transformation is only possible after one makes the following choice of volume element on the integration manifold:

$$\mathrm{d}\hat{R} = (p_1 - p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\theta, \tag{1.10}$$

whereas the would-be ‘natural’ choice of the non-negative volume element

$$\mathrm{d}\hat{R} = |p_1 - p_2|\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\theta,$$

as in (1.6), cannot yield a Gaussian function on the left-hand side of (1.1).

In the present paper we continue that study by considering two more specific cases, $O(2, 1)$ and $O(2, 2)$, and investigating in detail the validity of the Hubbard–Stratonovich transformation for the corresponding real hyperbolic domains. Note that for practical needs of the theory of disordered systems $O(2, 2)$ is the most important case related, in the supersymmetric version, to the basic object of the theory, the so-called two-point correlation function of resolvents of the random Schroedinger operator, see e.g. [1, 11].

In both $O(2, 1)$ and $O(2, 2)$ cases we are able to show that the naive choice of the measure (1.6) is never possible, but the Hubbard–Stratonovich transformation (1.1) can be saved provided we make a suitable alternative choice of $D\hat{P}$. These examples naturally suggest to put forward the following conjecture on the correct form of the Hubbard–Stratonovich transformation on a general $O(m, n)$-invariant Pruisken–Schäfer domain. Define

$$\hat{R} = \hat{T}^{-1}\hat{P}\hat{T}, \qquad \hat{P} = \operatorname{diag}(\hat{P}_1, \hat{P}_2) = \operatorname{diag}\,(p_{11}, \ldots, p_{1m}, p_{21}, \ldots, p_{2n}) \tag{1.11}$$

and the volume element

$$\mathcal{D}R = \mathrm{d}\mu_H(T)\mathcal{D}\hat{P}, \qquad \mathcal{D}\hat{P} = |\Delta[\hat{P}_1]|\cdot|\Delta[\hat{P}_2]|\prod_{i=1}^{m}\prod_{j=1}^{n}(p_{1i} - p_{2j}), \tag{1.12}$$

where $|\Delta[\hat{P}]|$ is the absolute value of the Vandermonde determinant and $\mathrm{d}\mu_H(\hat{T})$ stands for the invariant measure on $O(m, n)$. Further assume that the real matrix $\hat{A}$ is of the form $\hat{A} = \hat{A}_+\hat{L}$, where $\hat{A}_+$ is positive definite and $\hat{L}$ is the signature matrix $\hat{L}$ appearing in the definition of the pseudoorthogonal group $O(m,n)$[1]. Then the Hubbard–Stratonovich transformation over the Pruisken–Schäfer type of real hyperbolic domain is given by

$$\begin{aligned}\int \mathcal{D}\hat{R}\exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{iTr}\,\hat{R}\hat{A}\right)\\ &= \int_{-\infty}^{\infty}\mathcal{D}\hat{P}\exp\left(-\frac{1}{2}\left[\sum_{i=1}^{m}p_{1i}^2+\sum_{j=1}^{n}p_{2j}^2\right]\right)\int_{O(m,n)}\mathrm{d}\mu_H(\hat{T})\,\mathrm{e}^{-\mathrm{iTr}\hat{T}^{-1}\hat{P}\hat{T}\hat{A}}\\ &= \mathrm{const}\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\hat{A}^2}.\end{aligned} \tag{1.13}$$

Formula (1.13) is the central message of our work. The crucial difference of the choice (1.12) from the naive choice of the measure (1.6) is the absence of modulus for the factors $\prod_{i=1}^{m}\prod_{j=1}^{n}(p_{1i}-p_{2j})$. This forces the volume element to change sign inside the integration domain, in contrast to the conventional measures (densities) which are always positive as in, e.g., equation (1.3). Such a feature does not however in any way invalidate our Hubbard–Stratonovich formula, which should be interpreted as follows. The actual sign of $\mathcal{D}\hat{R}$ is determined by the inequalities between p_1's and p_2's. An ordered sequence of p_1's and p_2's thus defines a sub-domain of $\hat{R}$ on which the sign of $\mathcal{D}\hat{R}$ is fixed. Without loss of generality, we can assume $p_{11} > p_{12} > \cdots > p_{1m}$ and $p_{21} > p_{22} > \cdots > p_{2n}$. Then it is clear that the domain of integration in $\hat{R}$ is a union of altogether $(m+n)!/m!n!$ such disjoint sub-domains. Labelling a particular choice of the sub-domain of this sort by D_σ and defining $\mathrm{sgn}(\sigma)$ to be the sign of the volume element $\mathcal{D}\hat{R}$ on D_σ, the left-hand side of the integration formula we discuss is given by

$$\int \mathcal{D}\hat{R}\exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{iTr}\,\hat{R}\hat{A}\right) = \sum_{\sigma}\mathrm{sgn}(\sigma)\int_{D_\sigma}|\mathcal{D}\hat{R}|\exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{iTr}\,\hat{R}\hat{A}\right). \tag{1.14}$$

Interpreting our formula in this way, we always integrate over each sub-domain D_σ with the well-defined positive measures $|\mathcal{D}\hat{R}|$, but the lhs of equation (1.13) is given by an alternating sum of integrals on the disjoint sub-domains of $\hat{R}$. We believe that this coordinated change of sign is absolutely necessary to ensure the Gaussian form of the result of the integration, the conviction being based on the example of [13] and the results of the current paper.

We consider verification of this conjecture, as well as the discovery of a general mechanism which ensures its validity to be a challenging problem reserved for a future research[2].

2. Verification of the conjecture for the $O(2,1)$ case

In this section, we consider the Pruisken–Schäfer type of parameterization of integration domain (1.11) with $\hat{T}$ being an element of the real pseudoorthogonal group $O(2,1)$. The real matrix $\hat{A}$ in equation (1.13) is assumed to be of the form $\hat{A} = \hat{A}_+\hat{L}$, where $\hat{A}_+$ is positive definite and $\hat{L}$ is the signature matrix $\hat{L} = \mathrm{diag}(1,1,-1)$. As mentioned above, such matrices $\hat{A}$ can always be diagonalized as $\hat{A} = \hat{T}^{-1}\Lambda\hat{T}$, with $\hat{T}\in O(2,1)$ and Λ is a real diagonal matrix. By exploiting the invariance of the Haar measure we can safely choose $\hat{A}$ to be diagonal, as this choice obviously does not change the result of the integration.

[1] Such matrices can always be brought to a real diagonal form by $O(m,n)$ rotations, see e.g. appendix B of the paper [15].

[2] A method of proving the validity of the above conjecture in the general case $O(m,n)$ has recently been proposed by M R Zirnbauer and the present authors and will be published elsewhere [16].

Implementing the Pruisken–Schäfer parameterization, the integral on the right-hand side of equation (1.13) is of the form

$$I_{\mathrm{HS}}^{O(2,1)} = \int \mathcal{D}\hat{R} \exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A}\right)$$
$$= \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{3} p_i^2\right) \int_{O(2,1)} \mathrm{d}\mu(\hat{T})\,\mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{T}^{-1}\hat{P}\hat{T}\hat{A}}, \tag{2.1}$$

where $\hat{P} = \mathrm{diag}(p_1, p_2, p_3)$ and $\mathrm{d}\mu(\hat{T})$ is the invariant Haar measure on $O(2, 1)$. The crucial point is that we have to choose the volume element $\mathcal{D}\hat{P}$ to be, cf equation (1.12),

$$\mathcal{D}\hat{P} = |p_1 - p_2|(p_1 - p_3)(p_2 - p_3)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3. \tag{2.2}$$

We are going to demonstrate that it is only this choice that validates the Hubbard–Stratonovich transformation for our choice of the hyperbolic domain.

Note that the integral over the pseudoorthogonal group $O(2, 1)$ on the right-hand side of equation (2.1) is of the type of the Harish-Chandra–Itzykson–Zuber integral. Although integrals of this type have been known long ago for unitary groups [17] and extended more recently to pseudounitary groups [18], their analogues for (pseudo)orthogonal groups, which is relevant here, remains largely an open problem in mathematical physics, although a few interesting insights were obtained very recently [19, 20].

2.1. *Particular example of the $O(2, 1)$ Hubbard–Stratonovich transformation*

To elucidate main points of the calculation we first consider a special choice of the (diagonal) matrix $\hat{A}$, that is

$$\hat{A} = \mathrm{diag}(x, x, z) \Longrightarrow \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\hat{A}^2} = \mathrm{e}^{-\frac{1}{2}(2x^2+z)}. \tag{2.3}$$

Since $\hat{A}\hat{L} = \mathrm{diag}(x, x, -z) > 0$ according to our assumption, we have to require $x > 0 > z$.

The calculations will be simpler as such $\hat{A}$ effectively replaces the integration over the whole group $O(2, 1)$ with one over the non-compact Riemannian symmetric space

$$\frac{O(2,1)}{O(2)\times O(1)} \cong \frac{SO(2,1)}{S[O(2)\times O(1)]}. \tag{2.4}$$

Denote by $\mathrm{d}\mu(\hat{S})$ the $O(2, 1)$ invariant measure on the non-compact Riemannian symmetric space G/H, with $G = O(2, 1)$ and $H = O(2) \times O(1)$. For our special choice of the matrix $\hat{A}$ we obviously have

$$\int_{O(2,1)} \mathrm{d}\mu(\hat{T})\,\mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{T}^{-1}\hat{P}\hat{T}\hat{A}} = \int_{G/H} \mathrm{d}\mu(\hat{S})\,\mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}}, \tag{2.5}$$

so that equation (2.1) assumes the following form:

$$\int \mathcal{D}\hat{R}\,\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A}} = \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{3} p_i^2\right) \int_{G/H} \mathrm{d}\mu(\hat{S})\,\mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}}. \tag{2.6}$$

To perform the integration over the coset space G/H it is convenient to parameterize G/H with the projective coordinates (Z, Z^T). To this end, we introduce a 2×1 real matrix Z as

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{with the constraint} \quad 1 - Z^T Z \geqslant 0, \tag{2.7}$$

in terms of which the matrices $\hat{S}$ on G/H are given by

$$\hat{S} = \begin{pmatrix} (1-ZZ^T)^{-\frac{1}{2}} & Z(1-Z^TZ)^{-\frac{1}{2}} \\ Z^T(1-ZZ^T)^{-\frac{1}{2}} & (1-Z^TZ)^{-\frac{1}{2}} \end{pmatrix}. \tag{2.8}$$

It is direct to check that $\hat{S}^{-1}(Z, Z^T) = \hat{S}(-Z, -Z^T)$. The invariant measure $\mathrm{d}\mu(\hat{S})$ in projective coordinates can be calculated in the standard way [21] and is given by

$$\mathrm{d}\mu(\hat{S}) = \frac{\mathrm{d}Z\,\mathrm{d}Z^T}{(1-Z^TZ)^{\frac{3}{2}}}, \tag{2.9}$$

where $\mathrm{d}Z\,\mathrm{d}Z^T = \mathrm{d}z_1\,\mathrm{d}z_2$ and the integration domain is as specified in (2.7). Make the following change of variables:

$$\begin{cases} z_1 = r\cos\theta \\ z_2 = r\sin\theta, \end{cases} \qquad r \in [0, 1] \qquad \text{and} \qquad \theta \in [0, 2\pi]. \tag{2.10}$$

The integration on the right-hand side of equation (2.5) can be written as

$$\int_0^1 \frac{r\mathrm{d}r}{(1-r^2)^{\frac{3}{2}}} \int_0^{2\pi} \mathrm{d}\theta \exp\frac{\mathrm{i}}{2}\left\{\frac{r^2}{1-r^2}(x-z)(p_1-p_2)\cos 2\theta + \frac{x-z}{1-r^2}(p_1+p_2-2p_3)\right.$$
$$\left. + [x(p_1+p_2+2p_3)+z(p_1+p_2)]\right\}. \tag{2.11}$$

The integral over θ yields the standard Bessel functions in view of $\int_0^{\pi}\mathrm{d}\phi\,\mathrm{e}^{\mathrm{i}\beta\cos\phi} = \pi J_0(\beta)$, and introducing a new variable $t = \frac{r^2}{1-r^2}$, we rewrite (2.11) as

$$\mathrm{e}^{\mathrm{i}(x(p_1+p_2)+zp_3)} \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} J_0\left[\frac{t}{2}(x-z)(p_1-p_2)\right] \exp\left(\frac{\mathrm{i}t}{2}(x-z)(p_1+p_2-2p_3)\right). \tag{2.12}$$

Now we need to substitute equation (2.12) into the right-hand side of equation (2.6) and to integrate over $\hat{P}$, that is

$$I_{\mathrm{HS}}^{O(2,1)} = \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left\{-\frac{1}{2}\sum_{i=1}^{3} p_i^2 + i(x(p_1+p_2)+zp_3)\right.$$
$$\left. + \frac{\mathrm{i}t}{2}(x-z)(p_1+p_2-2p_3)\right\} J_0\left[\frac{t}{2}(x-z)(p_1-p_2)\right]. \tag{2.13}$$

After a straightforward, but lengthy calculation we arrive at the following result:

$$I_{\mathrm{HS}}^{O(2,1)} = \frac{\sqrt{2}\pi}{32} F[(x-z)^2]\,\mathrm{e}^{-\frac{1}{2}(2x^2+z^2)}, \tag{2.14}$$

where

$$F(a) = \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} \exp\left(-\frac{1}{2}(t^2+t)a\right)[1-a(2t^2+3t+1)]. \tag{2.15}$$

Note that the expression (2.14) already contains the Gaussian factor of precisely the form required by (2.3). Unfortunately, that factor is multiplied with a function $F[(x-z)^2]$ dependent on the combination $a = (x-z)^2$, the fact seemingly incompatible with the Hubbard–Stratonovich transformation. Miraculously enough, this factor is an a-independent constant! To verify this, we define $y = \sqrt{1+t}$, and carry out the integral explicitly:

$$\begin{aligned} F(a) &= \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \exp\left(-\frac{1}{2}(t^2+t)a\right)[1-a(2t^2+3t+1)] \\ &= \int_1^\infty \mathrm{d}y \exp\left(-\frac{a}{2}(y^4-y^2)\right)[1-a(2y^4-y^2)] \\ &= 1 - \lim_{y\to\infty} y \exp\left(-\frac{ay^2(y^2-1)}{2}\right) = 1. \end{aligned} \tag{2.16}$$

At the last step, we used the fact that a is strictly positive, as the case $a = 0$ should be excluded from the very beginning. Indeed, $a = 0$ implies $x = z$, contradicting to the original requirement $x > 0 > z$.

2.2. *General calculation for the $O(2, 1)$ case*

Now we are ready to present the complete proof of the Hubbard–Stratonovich transformation over $O(2, 1)$ domain. In the general case, we have $\hat{A} = \mathrm{diag}(x_1, x_2, z) = \hat{A}_1 + \hat{A}_2$ where $\hat{A}_1 = \mathrm{diag}(x, x, z)$ is the part considered in the previous example and $\hat{A}_2 = \mathrm{diag}(w, -w, 0)$. Here we defined the variables $x = (x_1 + x_2)/2$, $w = (x_1 - x_2)/2$. Our starting point is again equation (2.1), but we now have

$$\begin{aligned} I_{\mathrm{HS}}^{O(2,1)} &= \int \mathcal{D}\hat{R} \exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A}\right) \\ &= \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{3} p_i^2\right) \int_{G/H} \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}_1} \mathrm{d}\mu(\hat{S}) \int_H \mathrm{d}\mu(\hat{H})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}[\hat{H}\hat{A}_2\hat{H}^{-1}]}, \end{aligned} \tag{2.17}$$

where we assume $G = O(2, 1)$, $H = O(2) \times O(1)$ and $S = G/H$ as before.

The integration over H goes effectively over the group $SO(2)$ and the corresponding matrices can be parameterized in a standard way as $H = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$. Using the same parameters for the coset matrices $\hat{S}$ as in the previous section, we then find

$$\mathrm{Tr}\,\hat{S}^{-1}\hat{P}\hat{S}\hat{H}\hat{A}_2\hat{H}^{-1} = A\cos 2\phi + B\sin 2\phi, \tag{2.18}$$

where

$$\begin{aligned} A &= \frac{w}{4(1-r^2)}\big\{\big[(1+\sqrt{1-r^2})^2 + 2r^2\cos 2\theta + \cos 4\theta(1-\sqrt{1-r^2})^2\big]p_1 \\ &\quad + \big[2r^2\cos 2\theta - (1+\sqrt{1-r^2})^2 - \cos 4\theta(1-\sqrt{1-r^2})^2\big]p_2 - 4r^2\cos 2\theta p_3\big\} \\ B &= \frac{-w}{4(1-r^2)}\big\{\big[2r^2\sin 2\theta + \sin 4\theta(1-\sqrt{1-r^2})^2\big]p_1 \\ &\quad + \big[2r^2\sin 2\theta - \sin 4\theta(1-\sqrt{1-r^2})^2\big]p_2 - 4r^2\cos 2\theta p_3\big\}. \end{aligned} \tag{2.19}$$

The integration over ϕ is easily performed according to the formula

$$J_0(\sqrt{A^2+B^2}) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\phi \exp(\mathrm{i}\cos\phi A + \mathrm{i}\sin\phi B), \tag{2.20}$$

so that

$$\int_H \mathrm{d}\mu(\hat{H})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{H}\hat{A}_2\hat{H}^{-1}} = J_0(\sqrt{A^2+B^2}). \tag{2.21}$$

This should be inserted into equation (2.17), and remembering equations (2.11)–(2.13), we arrive at

$$I_{\mathrm{HS}}^{O(2,1)} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \int_0^{2\pi} \mathrm{d}\theta \exp\left\{ -\frac{1}{2}\sum_{i=1}^{3} p_i^2 + \mathrm{i}(x(p_1+p_2)+zp_3) \right.$$
$$\left. + \frac{\mathrm{i}t}{2}(x-z)(p_1+p_2-2p_3) + \frac{\mathrm{i}t}{2}(x-z)(p_1-p_2)\cos\theta \right\} J_0(\sqrt{A^2+B^2}), \qquad (2.22)$$

where again $\mathcal{D}\hat{P}$ is given by equation (2.2).

Note that variable 'w' responsible for the difference from the example considered in the previous section enters the formula only via the combination $\sqrt{A^2+B^2}$. A way of evaluating the above integral for $w \neq 0$ is to expand the Bessel function in Taylor series with the nth term proportional to w^{2n}, to integrate each term separately and then re-sum the series. A straightforward implementation of this program is however not immediate, and necessary steps of the proof are given in appendix A where it is shown that

$$I_{\mathrm{HS}}^{O(2,1)} = \mathrm{const} \exp\left[-x^2 - w^2 - \frac{z^2}{2} \right] = \mathrm{const} \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2 + z^2\right) \right], \qquad (2.23)$$

in precise agreement with the structure required by the Hubbard–Stratonovich transformation.

To summarize, we have demonstrated that for any $\hat{A} = \hat{T}_0 \,\mathrm{diag}(x_1, x_2, z)\hat{T}_0^{-1}$ and $\hat{T}_0 \in O(2,1)$ the identity

$$\int \mathcal{D}\hat{R} \exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A} \right) = \mathrm{const}\, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\hat{A}^2} \qquad (2.24)$$

holds, provided the volume element $\mathcal{D}P$ for the $\hat{P}$ integral is chosen in accordance with equation (2.2).

For the sake of comparison, one may try to repeat the above calculation with the 'naive' choice of measure $D\hat{P} = |\Delta(\hat{P})| \prod_{i=1}^{3} \mathrm{d}p_i$ instead of equation (2.2). We show in appendix B that such a choice invalidates the Hubbard–Stratonovich transformation. As another comparison, we also provide similar calculations in appendix C for the compact counterpart of this Pruisken–Schäfer domain corresponding to the group $O(3)$.

3. Results for the $O(2,2)$ case

In this section, we carry out the detailed calculation for the Hubbard–Stratonovich transformation over the $O(2,2)$ Pruisken–Schäfer domain. As the calculation turns out to be quite technically cumbersome, we restrict ourselves with the simplest non-trivial choice $\hat{A} = \mathrm{diag}(x, x, z, z)$, with $x > 0 > z$. Consequently, the integration domain $\hat{T} = O(2,2)$ effectively reduces to the non-compact Riemannian symmetric space (coset space)

$$\frac{O(2,2)}{O(2)\times O(2)} \simeq \frac{SO(2,2)}{S[O(2)\times O(2)]}. \qquad (3.1)$$

Parameterization of G/H, where $G = SO(2,2)$ and $H = S[O(2)\times O(2)]$, with the projective coordinates Z and Z^T is again in the form of equation (2.8) with Z and Z^T being real 2×2 matrices chosen in such a way ensuring that the matrix $1 - Z^T Z$ is positive definite:

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \quad \text{with} \quad 1 - Z^T Z \geqslant 0. \qquad (3.2)$$

We aim to prove the validity of the Hubbard–Stratonovich transformation with the Pruisken–Schäfer parameterization (1.5), where $T \in O(2,2)$ and $\hat{P} = \mathrm{diag}(p_1, p_2, p_3, p_4)$. To this end, we need to demonstrate that the following integral

$$
\begin{aligned}
I_{\mathrm{HS}}^{O(2,2)} &= \int \mathcal{D}\hat{R} \exp\left(-\frac{1}{2}\mathrm{Tr}\, \hat{R}^2 - \mathrm{i}\mathrm{Tr}\, \hat{R}\hat{A}\right) \\
&= \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{4} p_i^2\right) \int_{O(2,2)} \mathrm{d}\mu(\hat{T})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{T}^{-1}\hat{P}\hat{T}\hat{A}} \\
&= \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{4} p_i^2\right) \int_{G/H} \mathrm{d}\mu(\hat{S})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}}
\end{aligned} \tag{3.3}
$$

is, up to a constant factor, a product of Gaussian factors. The invariant measure $\mathrm{d}\mu(\hat{S})$ here is calculated in the standard way and is equal to [21]

$$
\mathrm{d}\mu(\hat{S}) = \frac{\mathrm{d}Z\, \mathrm{d}Z^T}{\det(1 - Z^T Z)^2}, \tag{3.4}
$$

where $\mathrm{d}Z\, \mathrm{d}Z^T = \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4$.

To carry out the integration over the coset space we introduce the polar coordinates parameterization for real matrices Z. This amounts to diagonalizing Z by two orthogonal rotations as

$$
Z = O_1 \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} O_2, \qquad \text{where} \quad r, s \in (-\infty, \infty), \qquad O_1, O_2 \in SO(2). \tag{3.5}
$$

A standard calculation (appendix D) shows that the Jacobian induced by changing variables from Z, Z^T to the polar coordinates is simply $|r^2 - s^2|$. We have accordingly

$$
\mathrm{d}Z\, \mathrm{d}Z^T = |r^2 - s^2|\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}\mu(O_1)\, \mathrm{d}\mu(O_2), \tag{3.6}
$$

where $\mathrm{d}\mu(O_1)$ and $\mathrm{d}\mu(O_2)$ are the invariant Haar measure of $SO(2)$. Using the polar coordinates the integral over coset space takes the form

$$
\begin{aligned}
&\int_{G/H} \mathrm{d}\mu(\hat{S})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}} \\
&\quad = \int D(r,s) \int_{SO(2)} \mathrm{d}\mu(O_1) \exp\left\{\mathrm{i}\mathrm{Tr}\left[O_1 \begin{pmatrix} \frac{x - zr^2}{1-r^2} & 0 \\ 0 & \frac{x - zs^2}{1-s^2} \end{pmatrix} O_1^{-1} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right]\right\} \\
&\qquad \times \int_{SO(2)} \mathrm{d}\mu(O_2) \exp\left\{\mathrm{i}\mathrm{Tr}\left[O_2^{-1} \begin{pmatrix} \frac{z - xr^2}{1-r^2} & 0 \\ 0 & \frac{z - xs^2}{1-s^2} \end{pmatrix} O_2 \begin{pmatrix} p_3 & 0 \\ 0 & p_4 \end{pmatrix}\right]\right\},
\end{aligned} \tag{3.7}
$$

where we denoted $D(r,s) = |r^2 - s^2| \mathrm{d}r\, \mathrm{d}s / (1 - r^2)^2 (1 - s^2)^2$.

The two integrals over $O(2)$ group manifold in equation (3.7) are easily carried out using the formula

$$
\begin{aligned}
&\int_{SO(2)} \mathrm{d}\mu(O) \exp\left\{\mathrm{i}\mathrm{Tr} O \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} O^{-1} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\right\} \\
&\quad = \exp\left[\frac{\mathrm{i}}{2}(a_1 + a_2)(b_1 + b_2)\right] J_0\left[\frac{1}{2}(a_1 - a_2)(b_1 - b_2)\right].
\end{aligned} \tag{3.8}
$$

Introducing at the next step the variables $u = \frac{1}{1-r^2}$ and $v = \frac{1}{1-s^2}$, we rewrite the resulting integral in equation (3.7) as

$$
e^{ix(p_3+p_4)+iz(p_1+p_2)}\int_1^\infty \frac{|u-v|\,du\,dv}{\sqrt{u(u-1)}\sqrt{v(v-1)}}\exp\left\{\frac{i}{2}(x-z)(p_1+p_2-p_3-p_4)(u+v)\right\}
\times J_0\left[\frac{1}{2}(x-z)(p_1-p_2)(u-v)\right]J_0\left[\frac{1}{2}(x-z)(p_3-p_4)(u-v)\right]. \quad (3.9)
$$

Now we have to perform the integration over variables in $\hat{P}$. As in the previous section, the crucial point is to choose the volume element $D\hat{P}$ in accordance with our main conjecture, that is

$$
D\hat{P} = |p_1-p_2|(p_1-p_3)(p_1-p_4)(p_2-p_3)(p_2-p_4)|p_3-p_4|\prod_{i=1}^{4} dp_i. \quad (3.10)
$$

The remaining steps are lengthy but straightforward. After a few variable changes we arrive at

$$
I_{\mathrm{HS}}^{SO(2,2)} = \frac{\pi}{128}\mathcal{F}[a]\exp[-x^2-z^2], \quad (3.11)
$$

with $a \equiv x - z$ and the function $\mathcal{F}[a]$ given in terms of a double integral as

$$
\mathcal{F}[a] = \int_1^\infty dt\,\exp\left(-\frac{a^2(t^2-1)}{4}\right)\int_0^{(t-1)^2}\frac{\frac{1}{4}a^4t^2(t^2-v)-a^2t^2+1}{\sqrt{[(t+1)^2-v][(t-1)^2-v]}}\,e^{-\frac{a^2v}{4}}\,dv. \quad (3.12)
$$

Integrating over v and defining a new variable $x = \frac{t+1}{2}$, we get

$$
\begin{aligned}
\mathcal{F}[a] = \frac{\pi}{128}\int_1^\infty dx\, e^{-a^2(x^2-x)}\frac{x-1}{x}\Bigg\{&[a^2(2x-1)^2-2]^2 \\
&\times\Phi_1\left[1,\frac{1}{2},\frac{3}{2},\left(\frac{x-1}{x}\right)^2,-a^2(x-1)^2\right] \\
&-\frac{8}{3}a^4(x-1)^2(2x-1)^2\Phi_1\left[2,\frac{1}{2},\frac{5}{2},\left(\frac{x-1}{x}\right)^2,-a^2(x-1)^2\right]\Bigg\},
\end{aligned} \quad (3.13)
$$

where Φ_1 is the degenerate hypergeometric series of two variables defined as [22]

$$
\Phi_1[\alpha,\beta,\gamma,x,y] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^my^n. \quad (3.14)
$$

From equation (3.11) we see that only if the factor $\mathcal{F}[a]$ is independent of its argument $a \equiv x - z$ the whole expression $I_{\mathrm{HS}}^{SO(2,2)}$ can be in the desired Gaussian form. It needs only a few lines of Maple or Mathematica code to numerically check that actually $\mathcal{F}[a] \equiv 1$, see figure 1.

Unfortunately, we were not able to find a way of verifying this miraculous identity analytically, as we managed to do in the previous case of $O(2,1)$ integral. Nevertheless, we do not think that the numerical data leave any doubt in the validity of our claim.

In conclusion, the above calculation shows that for $\hat{A} = \hat{T}_0\,\mathrm{diag}(x,x,z,z)\,\hat{T}_0^{-1}$ and $\hat{T}_0 \in O(2,2)$,

$$
\int \mathcal{D}\hat{R}\exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - i\mathrm{Tr}\,\hat{R}\hat{A}\right) = \mathrm{const}\,e^{-\frac{1}{2}\mathrm{Tr}\hat{A}^2}, \quad (3.15)
$$

provided measure for $\hat{P}$ integral is chosen to be equation (3.10).

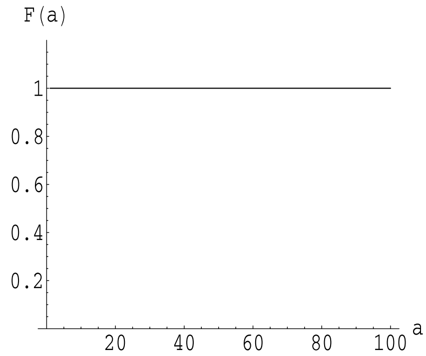

Figure 1. Function $F(a) = 1$ is a constant which does not depend on a.

It is again interesting to check what will be the result if we choose $\mathrm{d}\hat{P} = |\Delta(\hat{P})| \prod_{i=1}^{4} \mathrm{d}p_i$ instead of equation (3.10). It is shown in appendix E that this choice will make the Hubbard–Stratonovich identity invalid.

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Appendix A. Proof of equation (2.23)

Introducing the set of new variables

$$a = \tfrac{1}{2}(p_1 - p_2), \qquad b = \tfrac{1}{2}(p_1 + p_2), \qquad c = p_3 \tag{A.1}$$

and defining $t = \frac{r^2}{1-r^2}$, we can rewrite equation (2.22) as

$$\begin{aligned} I_{\mathrm{HS}}^{O(2,1)} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \int_0^\infty \mathrm{d}a \int_{-\infty}^{\infty} \mathrm{d}b\, \mathrm{d}c\, a[(b-c)^2 - a^2] \exp\left\{ -a^2 - b^2 - \frac{c^2}{2} \right. \\ \left. + \mathrm{i}[x(b+c) + zb] + \mathrm{i}\cos 2\theta\, ta(x-z) + \mathrm{i}(1+t)(x-z)(b-c) \right\} J_0(\sqrt{A^2+B^2}), \end{aligned} \tag{A.2}$$

where we have used equation (2.2). Next, we verify that

$$\begin{aligned} A^2 + B^2 &= w^2\{t^2(b-c)^2 + 2t(t+2)\cos 2\theta\, a(b-c) + a^2[t^2\cos^2 2\theta + 4(t+1)]\} \\ &= w^2\{[t(b-c) + a(t+2)\cos 2\theta]^2 + a^2[4(t+1)\sin^2 2\theta]\} \\ &= C^2 + D^2, \end{aligned} \tag{A.3}$$

where we defined

$$\begin{aligned} C &= w[t(b-c) + a(t+2)\cos 2\theta] \\ D &= 2wa\sqrt{t+1}\sin 2\theta. \end{aligned} \tag{A.4}$$

This allows us to write (cf (2.20))

$$J_0\left(\sqrt{A^2+B^2}\right) = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi\, \exp(\mathrm{i}\cos\phi C + \mathrm{i}\sin\phi D). \tag{A.5}$$

Using this representation of the Bessel function in equation (A.2) and defining $y = 1/\sqrt{1+t}$, we can readily carry out integrals over a, b, c and θ, and get

$$I_{\mathrm{HS}}^{O(2,1)} = \exp\left[-x^2 - \frac{z^2}{2}\right] F(w), \tag{A.6}$$

where we defined

$$F(w) = \int_1^{\infty} \mathrm{d}y \int_0^{\pi} \mathrm{d}\phi\, \exp\{-[w^2 - (x-z)^2]y^2 - [w\cos\phi(y^2-1) + (x-z)y^2]^2\}. \tag{A.7}$$

Note that although the above integral formally seems to depend on both w and $(x-z)$, we shall see below that it is a function of w only and is actually independent of the second combination.

To calculate $F(w)$ we find it convenient to apply first the standard Hubbard–Stratonovich transformation and 'linearize' the second term in the exponent by introducing an auxiliary Gaussian integral:

$$F(w) = \int_1^{\infty} \mathrm{d}y \int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}h\, \exp\{-[w^2 - (x-z)^2]y^2 - h^2 + 2\mathrm{i}h[w\cos\phi(y^2-1) + (x-z)y^2]\}. \tag{A.8}$$

Integration over ϕ yields the Bessel function which can be expanded in its Taylor series, and the Gaussian integral over h can be performed. In this way, we find

$$F(w) = \mathrm{const}\sum_{n=0}^{\infty} w^{2n} \int_1^{\infty} \mathrm{d}y\, \mathrm{e}^{-(x-z)^2(y^4-y^2)} \times\left\{\left[\frac{1}{y^2} - 2(1-2y^2)(x-z)^2\right]C_n + 2(1-2y^2)C_{n-1}\right\} \tag{A.9}$$

where we defined for $n \geqslant 0$

$$C_n = \sum_{m=0}^{n} \frac{(-)^{n-m}}{(n-m)!} y^{2(n-m)}(y^2-1)^{2m}\frac{(2m)!}{m!m!}\sum_{k=0}^{m}\frac{(-)^k}{4^k k!}\frac{[(x-z)y^2]^{2m-2k}}{(2m-2k)!} \tag{A.10}$$

and $C_n = 0$ for $n < 0$. In particular, the definition above implies

$$C_n\Big|_{y=1} = \frac{(-)^n}{n!}. \tag{A.11}$$

$F(w)$ can be found as we are now able to perform the integrations on the right-hand side of equation (A.9) as

$$\begin{aligned}
&\int_1^{\infty} \mathrm{d}y\, \mathrm{e}^{-(x-z)^2(y^4-y^2)}\left\{\left[\frac{1}{y^2} - 2(1-2y^2)(x-z)^2\right]C_n + 2(1-2y^2)C_{n-1}\right\} \\
&\quad = -\frac{\mathrm{e}^{-(x-z)^2(y^4-y^2)}}{y}\left[C_n + y^2(y^2-1)\sum_{i=0}^{n-1} a_{n,i}C_i\right]\Bigg|_{y=1}^{\infty} \\
&\quad = \frac{(-)^n}{n!}.
\end{aligned} \tag{A.12}$$

Here $a_{n,i}$'s are coefficients satisfying the following recursive relations:

$$a_{n,n-1} = \frac{2}{n}, \qquad \text{and} \qquad a_{n.i} = -\frac{1}{n}a_{n-1,i}, \qquad i = 0, \ldots, n-2, \quad a_{1,0} = 2. \tag{A.13}$$

In the last step of equation (A.12) we used the fact $x - z > 0$. We finally see that equation (A.12) implies the desired Gaussian expression

$$F(w) = \text{const}\, \mathrm{e}^{-w^2}. \tag{A.14}$$

Finally, substituting $F(w) \propto \mathrm{e}^{-w^2}$ back into equation (A.6) completes our proof of equation (2.23).

Appendix B. Calculation with the naive choice of the volume element $D\hat{P}$ for the $O(2,1)$ case

In this appendix, we show that the Hubbard–Stratonovich transformation for the $O(2,1)$ Pruisken–Schäfer domain is invalid if the volume element is chosen to be $D\hat{P} = |\Delta(\hat{P})| \prod_{i=1}^{3} \mathrm{d}p_i$.

Starting from equation (2.13), we make a change of integration variables as in equation (A.1). Then we write equation (2.13) as

$$\mathcal{I}_{\text{HS}}^{O(2,1)} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \int_{-\infty}^{\infty} \mathcal{D}\hat{P}$$
$$\times \exp\left(-a^2 - b^2 - \frac{c^2}{2} + \mathrm{i}(2xb + zc) + \mathrm{i}t(x-z)(b-c)\right) J_0\left[t(x-z)a\right], \tag{B.1}$$

where

$$D\hat{P} = 2|a((b-c)^2 - a^2)|\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c. \tag{B.2}$$

We rewrite the above integral as

$$\mathcal{I}_{\text{HS}}^{O(2,1)} = \mathcal{I}_{\text{HS},1}^{O(2,1)} + \mathcal{I}_{\text{HS},2}^{O(2,1)}, \tag{B.3}$$

where we defined

$$\mathcal{I}_{\text{HS},1}^{O(2,1)} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \int_{-\infty}^{\infty} 2|a|((b-c)^2 - a^2)\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c$$
$$\times \exp\left(-a^2 - b^2 - \frac{c^2}{2} + \mathrm{i}(2xb + zc) + \mathrm{i}t(x-z)(b-c)\right) J_0\left[t(x-z)a\right] \tag{B.4}$$

and

$$\mathcal{I}_{\text{HS},2}^{O(2,1)} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1+t}} \int_0^{|b-c|} 4|a|(a^2 - (b-c)^2)\, \mathrm{d}a \int_{-\infty}^{\infty} \mathrm{d}b\, \mathrm{d}c$$
$$\times \exp\left(-a^2 - b^2 - \frac{c^2}{2} + \mathrm{i}(2xb + zc) + \mathrm{i}t(x-z)(b-c)\right) J_0\left[t(x-z)a\right]. \tag{B.5}$$

Let us stress that it is the contribution $\mathcal{I}_{\text{HS},2}^{O(2,1)}$ which encapsulates the difference between the definition $D\hat{P} = |\Delta(\hat{P})| \prod_{i=1}^{3} \mathrm{d}p_i$ which is positive definite and equation (2.2) which is sign indefinite. Such a term has cancelled out when the volume element was chosen to be equation (2.2). The first contribution $\mathcal{I}_{\text{HS},1}^{O(2,1)}$ is nothing but $I_{\text{HS}}^{O(2,1)}$ as calculated in section 2, and we proved that it in the Gaussian form

$$\mathcal{I}_{\text{HS}}^{O(2,1)} = \text{const} \exp\left[-\frac{1}{2}(2x^2 + z^2)\right] + \mathcal{I}_{\text{HS},2}^{O(2,1)}. \tag{B.6}$$

In the remaining part of this appendix we will demonstrate that the first contribution $\mathcal{I}_{\mathrm{HS},2}^{O(2,1)}$ is not in a Gaussian form, thus invalidating the Hubbard–Stratonovich transformation.

Define $m = b + c$, $n = b - c$ and integrate over m. We get

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,1)} = \frac{2\sqrt{6\pi}}{3} \exp\left[-\frac{1}{6}(2x+z)^2\right] \int_{-\infty}^{\infty} \mathrm{d}n\, n^4 \exp\left\{-\left(\frac{1}{3}+a^2\right)n^2 + in(x-z)\left(\frac{2}{3}+t\right)\right\}$$
$$\times \int_0^1 \mathrm{d}a \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} a(a^2-1) J_0\Big[t(x-z)na\Big]. \tag{B.7}$$

It is clear that $\mathcal{I}_{\mathrm{HS},2}^{O(2,1)}$ will be in the desired Gaussian form if the integral part of the above formula is $\propto \exp(-(x-z)^2/3)$. To check this, it is sufficient to consider a special case $x \to z$, i.e. $|x - z| \ll 1$. In this limit, we can approximate the integral by setting the argument of the Bessel function in the integrand to zero. This gives

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,1)} \propto \mathrm{e}^{-\frac{(2x+z)^2}{6}} \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} \int_{-\infty}^{\infty} \mathrm{d}n\, n^4 \exp\left\{-\frac{1}{3}n^2 + \mathrm{i}n(x-z)\left(\frac{2}{3}+t\right)\right\}$$
$$\times \int_0^1 \mathrm{d}a\, a(a^2-1)\exp(-n^2a^2). \tag{B.8}$$

The integral over a is simply

$$\int_0^1 \mathrm{d}a\, a(1-a^2)\exp(-n^2a^2) = \frac{1}{2n^4}[\exp(-n^2) + n^2 - 1]. \tag{B.9}$$

Carrying out the standard Gaussian integrals over n, we get

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,1)} \propto \mathrm{e}^{-\frac{(2x+z)^2}{6}} \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{1+t}} \left[\frac{1}{2}[(x-z)^2(3t+2)^2 - 2]\exp\left(-\frac{1}{12}(x-z)^2(3t+2)^2\right)\right.$$
$$\left. - \exp\left(-\frac{1}{48}(x-z)^2(3t+2)^2\right)\right]. \tag{B.10}$$

The integral over t is divergent if $x - z = 0$, as expected, and in the limit $|x - z| \ll 1$ it is a well-defined expression dominated by $t \sim (x-z)^{-1} \gg 1$ so that $\mathcal{I}_{\mathrm{HS},2}^{\mathrm{O}(2,1)} \sim (x-z)^{-1/2}$. Such a pre-exponential factor clearly precludes the expression to be in the desired Gaussian form.

Appendix C. Calculations for the standard case $O(3)$

In this appendix, we repeat calculations similar to those in section 2 and appendix B, but this time for the compact case of $O(3)$ group. Although the Hubbard–Stratonovich transformation for $O(3)$ symmetry is trivially valid in the original formulation, it is instructive to have a comparison between $O(3)$ and $O(2,1)$ in the polar representation, as it helps to understand peculiarities of the non-compact case.

First, we consider an integral similar to equation (2.1), with integration of $\hat{T}$ going this time over $O(3)$ instead of $O(2,1)$. We consider only the simplest case setting $\hat{A} = \mathrm{diag}(x, x, z)$ and deal with the following integral:

$$I_{\mathrm{HS}}^{O(3)} = \int \mathcal{D}\hat{R} \exp\left(-\frac{1}{2}\mathrm{Tr}\,\hat{R}^2 - \mathrm{i}\mathrm{Tr}\,\hat{R}\hat{A}\right)$$
$$= \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left(-\frac{1}{2}\sum_{i=1}^{3} p_i^2\right) \int_{G/H} \mathrm{d}\mu(\hat{S})\, \mathrm{e}^{-\mathrm{i}\mathrm{Tr}\hat{S}^{-1}\hat{P}\hat{S}\hat{A}}, \tag{C.1}$$

where $G = O(3)$ and $H = O(2) \times O(1)$. Elements of this compact coset is parameterized as

$$s = g_H = \begin{pmatrix} (1 + ZZ^T)^{-\frac{1}{2}} & Z(1 + Z^T Z)^{-\frac{1}{2}} \\ Z^T(1 + ZZ^T)^{-\frac{1}{2}} & (1 + Z^T Z)^{-\frac{1}{2}} \end{pmatrix}, \tag{C.2}$$

where we introduced the 2×1 real matrix Z as the convenient coordinate on G/H, with

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \text{with} \quad z_1 \text{ and } z_2 \text{ arbitrary real.} \tag{C.3}$$

Similar to the non-compact case, $s^{-1}(Z, Z^T) = s(-Z, -Z^T)$. The invariant measure $\mathrm{d}\mu(\hat{S})$ in the projective coordinates is given by

$$\mathrm{d}\mu(\hat{S}) = \frac{\mathrm{d}Z\, \mathrm{d}Z^T}{(1 + Z^T Z)^{\frac{3}{2}}}, \qquad \text{where} \qquad \mathrm{d}Z\, \mathrm{d}Z^T = \mathrm{d}z_1\, \mathrm{d}z_2. \tag{C.4}$$

The integration over the coset is now straightforward and calculations are done parallel to those in section 2. After some algebra and a few changes of variables, we get

$$I_{\mathrm{HS}}^{O(3)} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \int_{-\infty}^{\infty} \mathcal{D}\hat{P} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{3} p_i^2 + \mathrm{i}(x(p_1 + p_2) + z p_3) + \frac{\mathrm{i}t}{2}(z - x)(p_1 + p_2 - 2p_3) \right\} J_0\left[\frac{t}{2}(x - z)(p_1 - p_2) \right]. \tag{C.5}$$

The difference between equations (C.5) and (2.13) is due to the difference between compact and non-compact integration manifolds.

A crucial difference in the $O(3)$ case is that the volume elements $\mathcal{D}\hat{P}$ in the above formula is $\mathcal{D}\hat{P} = |\Delta(\hat{P})| \prod_{i=1}^{3} \mathrm{d}p_i$, instead of equation (2.2). We have seen in appendix B that this choice of $\mathcal{D}\hat{P}$ when applied for $O(2, 1)$ symmetry would yield a form which is not Gaussian. In the remaining part of this appendix we show that in the case of $O(3)$ the result is in contrast to Gaussian.

Define the same set of integration variables as equation (A.1) and use them in equation (C.5). We have

$$\mathcal{I}_{\mathrm{HS}}^{O(3)} = \mathcal{I}_{\mathrm{HS},1}^{O(3)} + \mathcal{I}_{\mathrm{HS},2}^{O(3)}, \tag{C.6}$$

where we defined

$$\mathcal{I}_{\mathrm{HS},1}^{O(3)} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \int_{-\infty}^{\infty} 2|a|((b-c)^2 - a^2)\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c \times \exp\left(-a^2 - b^2 - \frac{c^2}{2} + \mathrm{i}(2xa + zc) - \mathrm{i}t(x - z)(a - c) \right) J_0[t(x - z)a] \tag{C.7}$$

and

$$\mathcal{I}_{\mathrm{HS},2}^{O(3)} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \int_0^{|b-c|} 4|a|(a^2 - (b-c)^2)\, \mathrm{d}a \int_{-\infty}^{\infty} \mathrm{d}b\, \mathrm{d}c \times \exp\left(-a^2 - b^2 - \frac{c^2}{2} + \mathrm{i}(2xa + zc) - \mathrm{i}t(x - z)(a - c) \right) J_0[t(x - z)a]. \tag{C.8}$$

Note again that $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$ corresponds to the definition (2.2) and $\mathcal{I}_{\mathrm{HS},2}^{O(3)}$ emerges only because the volume element is positive definite in the current case.

First, we deal with $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$. Carrying out simple Gaussian integrations over a, b and c we find

$$\mathcal{I}_{\mathrm{HS},1}^{O(3)} = \frac{\sqrt{2}\pi}{32} \mathcal{F}_1(x, z) \exp\left(-\frac{1}{2}(2x^2 + z^2) \right), \tag{C.9}$$

where

$$\mathcal{F}_1(x,z) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \exp\left(-\frac{1}{2}(t^2+t)(x-z)^2\right)\left[1-(x-z)^2(2t^2+3t+1)\right]. \tag{C.10}$$

Using $a=(x-z)^2$ and $y=\sqrt{1-t}$ we immediately see that

$$\begin{aligned}\mathcal{F}_1(a) &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \exp\left(-\frac{1}{2}(t^2+t)(x-z)^2\right)[1-(x-z)^2(2t^2-3t+1)]\\ &= \int_0^1 \mathrm{d}y \exp\left(-\frac{a}{2}(y^4-y^2)\right)[1-a(2y^4-y^2)]\\ &= 1-\lim_{y\to 0} y\exp\left(-\frac{ay^2(y^2-1)}{2}\right) = 1.\end{aligned} \tag{C.11}$$

Here, the integral over y is the same as that in equation (2.16) with different upper and lower limits, but the result is the same. This completes our proof that $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$ is indeed in the Gaussian form. We also note there is certain kind of duality between $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$ and $\mathcal{I}_{\mathrm{HS}}^{O(2,1)}$.

As we know already, the integral $\mathcal{I}_{\mathrm{HS}}^{O(3)}$ is of the Gaussian form $\propto \exp[-x^2-z^2/2]$, and we have just shown that the same holds for $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$; the second terms $\mathcal{I}_{\mathrm{HS},2}^{O(3)}$ can then only be either 0 or the same Gaussian form as $\mathcal{I}_{\mathrm{HS},1}^{O(3)}$. To see which is the case, it is sufficient to consider the same limit $x\to z$ as we did in appendix B. In the limit $|x-z|\ll 1$, we find

$$\begin{aligned}\mathcal{I}_{\mathrm{HS},2}^{O(3)} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} \int_0^{|b-c|} 4|a|(a^2-(b-c)^2)\,\mathrm{d}a \int_{-\infty}^{\infty} \mathrm{d}b\,\mathrm{d}c\\ \times\exp\left(-a^2-b^2-\frac{c^2}{2}+\mathrm{i}(2xa+zc)\right).\end{aligned} \tag{C.12}$$

One can perform all the integrations in this formula explicitly, and show that

$$\mathcal{I}_{\mathrm{HS},2}^{O(3)} \propto \exp\left\{-\frac{1}{2}(2x^2+z^2)\right\} \tag{C.13}$$

as expected.

Appendix D. Jacobian of the transformation from Z to polar coordinates

Write the polar coordinates decomposition in equation (3.5) as $Z = O_1\Lambda O_2$, where $\Lambda = \begin{pmatrix} r & 0\\ 0 & s\end{pmatrix}$. We have

$$\begin{cases}\mathrm{d}Z = \mathrm{d}O_1\Lambda O_2 + O_1\mathrm{d}\Lambda O_2 + O_1\Lambda\,\mathrm{d}O_2\\ \mathrm{d}Z^T = \mathrm{d}O_2^T\Lambda O_1 + O_2^T\mathrm{d}\Lambda O_1 + O_2^T\Lambda\,\mathrm{d}O_1.\end{cases} \tag{D.1}$$

Following the standard way of derivation, see e.g. [21], we have

$$\begin{aligned}\mathrm{d}^2S = \mathrm{Tr}\,\mathrm{d}Z\,\mathrm{d}Z^T = \mathrm{Tr}\big\{&O_1^T\mathrm{d}O_1\Lambda O_2\mathrm{d}O_2^T\Lambda + \Lambda\,\mathrm{d}\Lambda O_2\mathrm{d}O_2^T + \Lambda^2\mathrm{d}O_2\mathrm{d}O_2^T\\ &+O_1^T\mathrm{d}O_1\Lambda\,\mathrm{d}\Lambda + d^2\Lambda + \Lambda\,\mathrm{d}\Lambda\,\mathrm{d}O_2O_2^T\\ &+\Lambda^2\mathrm{d}O_1^T\mathrm{d}O_1 + \Lambda\,\mathrm{d}\Lambda\,\mathrm{d}O_1^TO_1 + \Lambda\,\mathrm{d}O_2O_2^T\Lambda\,\mathrm{d}O_1^TO_1\big\}.\end{aligned} \tag{D.2}$$

Next we define

$$\begin{cases}\delta O_1 = O_1^T\mathrm{d}O_1\\ \delta O_2 = \mathrm{d}O_2O_2^T.\end{cases} \tag{D.3}$$

Recalling that δO_1 and δO_2 are skew-symmetric matrices, they can be written as

$$\delta O_1 = \begin{pmatrix} 0 & \delta O_{1,12} \\ -\delta O_{1,12} & 0 \end{pmatrix}, \qquad \delta O_2 = \begin{pmatrix} 0 & \delta O_{2,12} \\ -\delta O_{2,12} & 0 \end{pmatrix}. \tag{D.4}$$

We find

$$\begin{aligned} d^2 S &= \mathrm{Tr}\{d^2\Lambda - \Lambda^2 \delta O_1 \delta O_1 - \Lambda^2 \delta O_2 \delta O_2 - 2\Lambda \delta O_1 \Lambda \delta O_2\} \\ &= (\delta O_{1,12}, \delta O_{2,12}, \mathrm{d}r, \mathrm{d}s) \begin{pmatrix} r^2+s^2 & 2rs & & \\ 2rs & r^2+s^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \delta O_{1,12} \\ \delta O_{2,12} \\ \mathrm{d}r \\ \mathrm{d}s \end{pmatrix} \\ &= \mathrm{d}x^i g_{ij} \, \mathrm{d}x^j. \end{aligned} \tag{D.5}$$

In the last step the summation over repeated indices is assumed. Jacobian is then given by

$$\text{Jacobian} = \sqrt{\det g} = |r^2 - s^2|. \tag{D.6}$$

Appendix E. Calculation with the alternative volume element $D\hat{P}$ for the $O(2,2)$ case

In this appendix, we calculate equation (3.3) with the volume element $D\hat{P} = |\Delta[\hat{P}]| \prod_{i=1}^4 \mathrm{d}p_i$ used instead of equation (3.10). We show that by this choice the final result is not in the Gaussian form, hence the corresponding Hubbard–Stratonovich transformation cannot be valid.

First we redefine the integration variables

$$\begin{cases} a = \frac{1}{2}(p_1 + p_2) \\ b = \frac{1}{2}(p_1 - p_2) \\ c = \frac{1}{2}(p_3 + p_4) \\ d = \frac{1}{2}(p_3 - p_4). \end{cases} \tag{E.1}$$

Then we have

$$|\Delta[\hat{P}]| = 4|bd| \cdot |[(a-c+d)^2 - b^2][(a-c-d)^2 - b^2]|. \tag{E.2}$$

Use equation (3.9) and $D\hat{P}$ defined above to find

$$\begin{aligned} \mathcal{I}_{\mathrm{HS}}^{O(2,2)} = \int_1^\infty \frac{|u-v| \mathrm{d}u \, \mathrm{d}v}{\sqrt{u(u-1)}\sqrt{v(v-1)}} \int D\hat{P} \, \mathrm{e}^{-a^2-b^2-c^2-d^2+\mathrm{i}(x-z)(a-c)(u+v)} \\ \times J_0[b(x-z)(u-v)] J_0[d(x-z)(u-v)]. \end{aligned} \tag{E.3}$$

As in appendix B we can split $\mathcal{I}_{\mathrm{HS}}^{O(2,2)}$ into two parts:

$$\mathcal{I}_{\mathrm{HS}}^{O(2,2)} = \mathcal{I}_{\mathrm{HS},1}^{O(2,2)} + \mathcal{I}_{\mathrm{HS},2}^{O(2,2)}. \tag{E.4}$$

Here, the contribution $\mathcal{I}_{\mathrm{HS},1}^{O(2,2)}$ is precisely $I_{\mathrm{HS}}^{O(2,2)}$ we calculated in section 3, and we know it is in a Gaussian form. It is the second contribution, $\mathcal{I}_{\mathrm{HS},2}^{O(2,2)}$, which arises from the difference between the two definitions of $D\hat{P}$, and it is given by

$$\begin{aligned} \mathcal{I}_{\mathrm{HS},2}^{O(2,2)} \propto \int_{-\infty}^\infty \mathrm{d}a \, \mathrm{d}c \int_0^\infty \mathrm{d}d \int_{||a-c|-d|}^{|a-c|+d} \mathrm{d}b \, bd[(a-c+d)^2 - b^2][(a-c-d)^2 - b^2] \\ \times \int_1^\infty \frac{|u-v| \mathrm{d}u \, \mathrm{d}v}{\sqrt{u(u-1)}\sqrt{v(v-1)}} \mathrm{e}^{-a^2-b^2-c^2-d^2+2\mathrm{i}xc+2\mathrm{i}za+\mathrm{i}(x-z)(a-c)(u+v)} \\ \times J_0[b(x-z)(u-v)] J_0[d(x-z)(u-v)]. \end{aligned} \tag{E.5}$$

In the remaining part of this section we demonstrate that $\mathcal{I}_{\mathrm{HS},2}^{O(2,2)}$ is not in the Gaussian form, thus $\mathcal{I}_{\mathrm{HS}}^{O(2,2)}$ is not either. Which means that the Hubbard–Stratonovich transformation fails with this different choice of $D\hat{P}$.

First, we define $m = a + c$ and $n = a - c$. It is clear that the integral over m is decoupled from other integrations and can be easily performed. Again, it is sufficient to consider the limit $|x - z| \ll 1$. For the same reason as in appendix B, we set the two Bessel terms to be 1 in this limit. We then have

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,2)} \propto \exp\left\{-\frac{1}{2}(x+z)^2\right\} \int_0^\infty \mathrm{d}n \int_0^\infty \mathrm{d}d \int_{|n-d|}^{n+d} \mathrm{d}b\, bd[(n+d)^2 - b^2][(n-d)^2 - b^2]$$
$$\times \int_1^\infty \frac{|u-v|\mathrm{d}u\,\mathrm{d}v}{\sqrt{u(u-1)}\sqrt{v(v-1)}} \exp\{-n^2 - b^2 - d^2\} \cos[n(x-z)(u+v-1)]. \quad \text{(E.6)}$$

The integral part of the above formula needs to be $\propto \exp\left\{-\frac{1}{2}(x-z)^2\right\}$ in order to make $\mathcal{I}_{\mathrm{HS},2}^{O(2,2)}$ Gaussian. The remaining calculations are lengthy but direct. We perform Gaussian-type integrals over b, d and n then define new integration variables $X = u + v - 1$ and $Y = u - v$. After integrating over Y we get

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,2)} \propto \int_0^\infty \mathrm{d}X \ln(2X+1)\left\{8 - 2a^2X^2 + \sqrt{\pi}\exp\left(-\frac{a^2X^2}{4}\right) aX(a^2X^2 - 6)\mathrm{Erfi}\left[\frac{aX}{2}\right]\right\}, \quad \text{(E.7)}$$

where we defined $a = x - z$, and Erfi stands for the error function of imaginary argument. Integrating by parts we bring the above integral to the form

$$\mathcal{I}_{\mathrm{HS},2}^{O(2,2)} \propto \int_0^\infty \mathrm{d}X \frac{1}{2X+1}\left(\frac{aX}{4} - \frac{\sqrt{\pi}}{8}\exp\left(-\frac{a^2X^2}{4}\right)(a^2X^2 - 2)\mathrm{Erfi}\left[\frac{aX}{2}\right]\right). \quad \text{(E.8)}$$

Again, in the limit $|x-z| \ll 1$ the integral over X is dominated by the region $X \sim (x-z)^{-1} \gg 1$. Changing variable $aX \to X$ and expanding in terms of $|x-z|$, we find to the lowest order, $\mathcal{I}_{\mathrm{HS},2}^{O(2,2)} = c_0 - |x-z|c_1 + O((x-z)^2)$, with c_0 and c_1 being some constants. In this way one finds that $\mathcal{I}_{\mathrm{HS},2}^{O(2,2)}$ is a function of $(x-z)$, but is clearly not in the Gaussian form. So we conclude that $\mathcal{I}_{\mathrm{HS}}^{O(2,2)}$ is not Gaussian.

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